Fully Homomorphic Encryption

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ENEE-459C

Homomorphic encryption

In the beginning, there was *symmetric* encryption.

If you had the key you could encrypt …

If you had the key you could encrypt ...

Message:	**ATTACK AT DAWN**
Key: +3	↓↓↓↓↓↓ ↓↓ ↓↓↓↓
Ciphertext:	**DWWDFN DW GDZQ**

If you had the key you could decrypt …

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Message:	**ATTACK AT DAWN**
Key: +3	↑↑↑↑↑↑ ↑↑ ↑↑↑↑
Ciphertext:	**DWWDFN DW GDZQ**

… and some people were happy.

Then, there was asymmetric encryption.

Some people encrypted …

Some people encrypted …

… others decrypted.

Internet took off…

Internet took off…

… and more people were happy.

The first and most used asymmetric cipher was RSA.

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$$E(m) = m^e \pmod{n}$$

Some people noticed the algebraic structure …

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$$E(m_1) = {m_1}^e \qquad E(m_2) = {m_2}^e$$

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$$E(m_1) \times E(m_2)$$

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$$E(m_1) = {m_1}^e \qquad E(m_2) = {m_2}^e$$

Multiply ciphertexts

$$E(m_1) \times E(m_2)$$
$$= {m_1}^e \times {m_2}^e$$

Some people noticed the algebraic structure …

$$E(m_1) = {m_1}^e \qquad E(m_2) = {m_2}^e$$

Ergo …

$$\begin{aligned} & E(m_1) \times E(m_2) \\ & \quad = {m_1}^e \times {m_2}^e \\ & \quad = (m_1 \times m_2)^e \end{aligned}$$

Some people noticed the algebraic structure …

$$E(m_1) = m_1{}^e \qquad E(m_2) = m_2{}^e$$

Ergo …

$$\begin{aligned} E(m_1) \times E(m_2) & \\ &= m_1{}^e \times m_2{}^e \\ &= (m_1 \times m_2)^e \\ &= E(m_1 \times m_2) \end{aligned}$$

People thought …

… if only RSA worked additively …

People mused …

… if only RSA worked additively …

we could compute sums …

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we could compute sums …

and averages …

People mused …

… if only RSA worked additively …

we could compute sums …

and averages …

and tally elections …

An additive encryption homomorphism …

An additive encryption homomorphism …

$$E(m, r) = r^e c^m$$

$$E(m_1, r_1) = {r_1}^e c^{m_1} \qquad E(m_2, r_2) = {r_2}^e c^{m_2}$$

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$$E(m_1, r_1) \times E(m_2, r_2)$$

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$$E(m_1, r_1) \times E(m_2, r_2)$$
$$= {r_1}^e c^{m_1} \times {r_2}^e c^{m_2}$$

$$E(m_1, r_1) = r_1{}^e c^{m_1} \qquad E(m_2, r_2) = r_2{}^e c^{m_2}$$

$$\begin{aligned} E(m_1, r_1) \times E(m_2, r_2) \\ &= r_1{}^e c^{m_1} \times r_2{}^e c^{m_2} \\ &= (r_1 r_2)^e c^{m_1 + m_2} \end{aligned}$$

$$E(m_1, r_1) = \ {r_1}^e c^{m_1} \qquad E(m_2, r_2) = \ {r_2}^e c^{m_2}$$

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The product of encryptions of two messages is *an* encryption of the sum of the two messages.

What people really wanted was the ability to do arbitrary computing on encrypted data…

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… and this required the ability to compute *both* sums *and* products …

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… and this required the ability to compute *both* sums *and* products …

… on the same data set!

People tried to do this for years …

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… and years …

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… and years …

… and years …

People tried to do this for years …

… and years …

… and years …

… with no success.

WHY does ADD AND MULTIPLY help?

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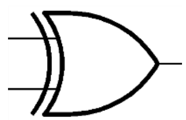
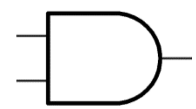


XOR (add mod 2)

0 XOR 0	0
1 XOR 0	1
0 XOR 1	1
1 XOR 1	0



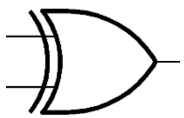
AND (mult mod 2)

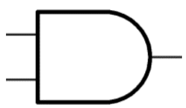
0 AND 0	0
1 AND 0	0
0 AND 1	0
1 AND 1	1

WHY does ADD AND MULTIPLY help?

… because {XOR,AND} is Turing-complete …

(any function can be written as a combination of XOR and AND gates)





XOR (add mod 2)

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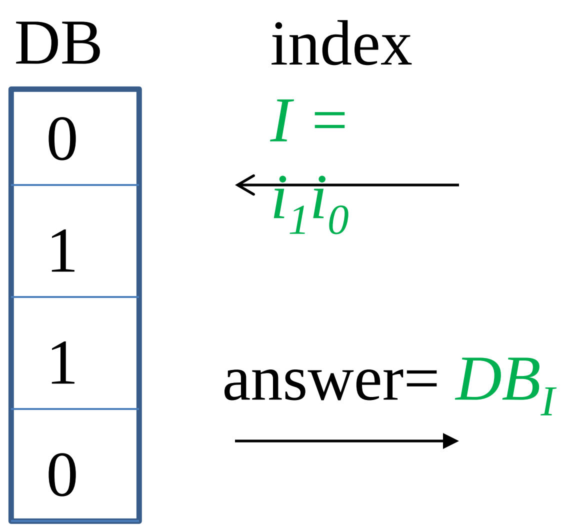
Example: Searching a database



WHY does ADD AND MULTIPLY help?

… because {XOR,AND} is Turing-complete …

(any function can be written as a combination of XOR and AND gates)

Example: Searching a database

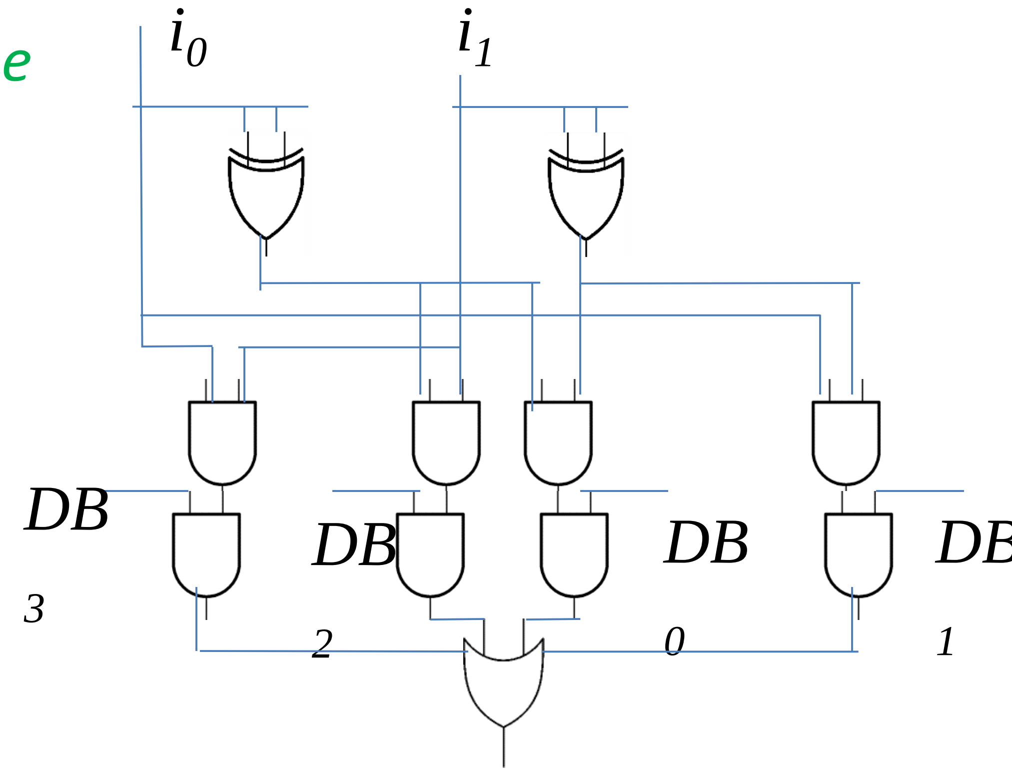
DB

0
1
1
0

index

$I = i_1 i_0$

answer= DB_I

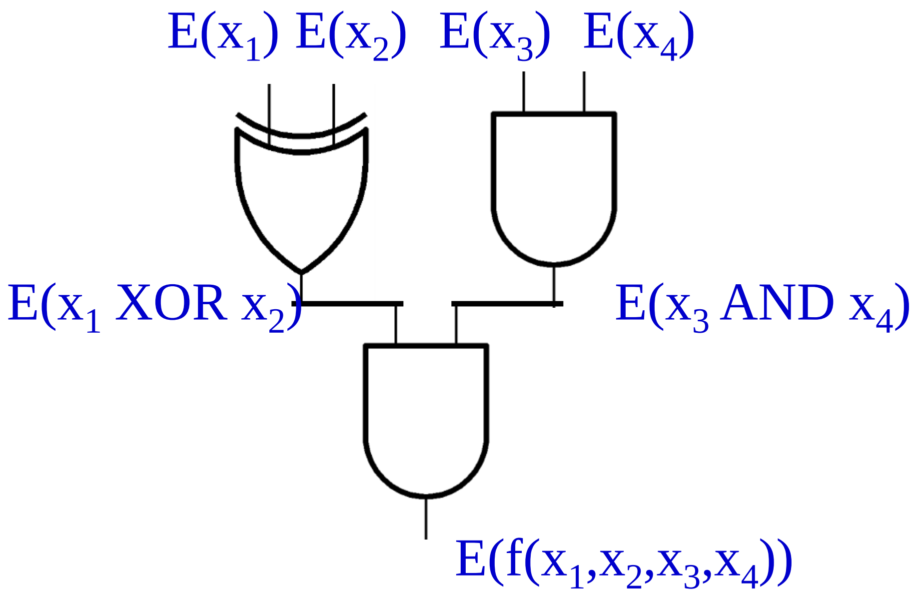


WHY does ADD AND MULTIPLY help?

… because {XOR,AND} is Turing-complete …

… if you can compute XOR and AND on encrypted bits…

… you can compute ***ANY*** function on encrypted inputs…



This is A M A Z I N G!

Private bing Search

Private Cloud computing

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Private bing Search

Private Cloud computing

In general,

Delegate *processing* of data

without giving away *access* to it

People tried to compute both AND and XOR on encrypted bits …

… for years …

… and years …

… with no success.

Well, actually, there were some ***partial*** answers …

Fully homomorphic

Josh's system

MANY add
ZERO mult

MANY add
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Well, actually, there were some ***partial*** answers …

Fully homomorphic

Josh's system

Boneh,Goh & Nissim

MANY add
ZERO mult

MANY add
1 mult

MANY add
MANY mult

… and some bold attempts [Fellows-Koblitz] …

… which were quickly broken …

Fully homomorphic

Josh's system

Boneh,Goh & Nissim

MANY add
ZERO mult

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1 mult

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… until, in October 2008 …

... until, in October 2008 ...

... ***Craig Gentry*** came up with the first

fully homomorphic encryption scheme ...

How does it work?

What is the magic?

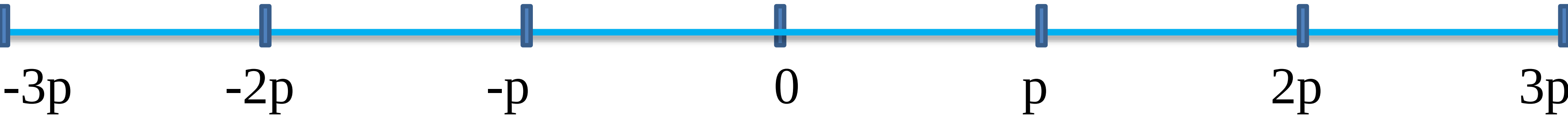
Gentry's scheme was complex …

… it used advanced algebraic number theory …

Some of us asked: can we make this really simple? …

TODAY: Secret-key (Symmetric-key) Encryption

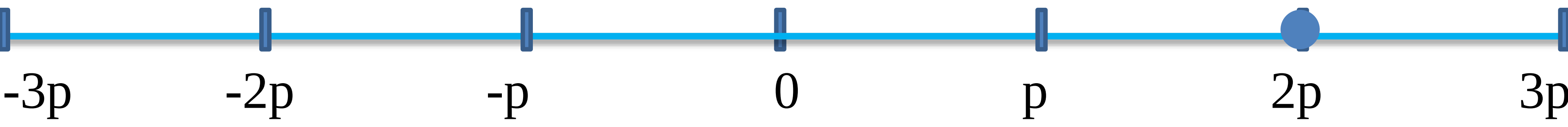
Secret key: large *odd* number **p**



Secret key: large *odd* number **p**

To Encrypt a bit ***b*****:**

– pick a (random) "large" multiple of p, say **q·p**



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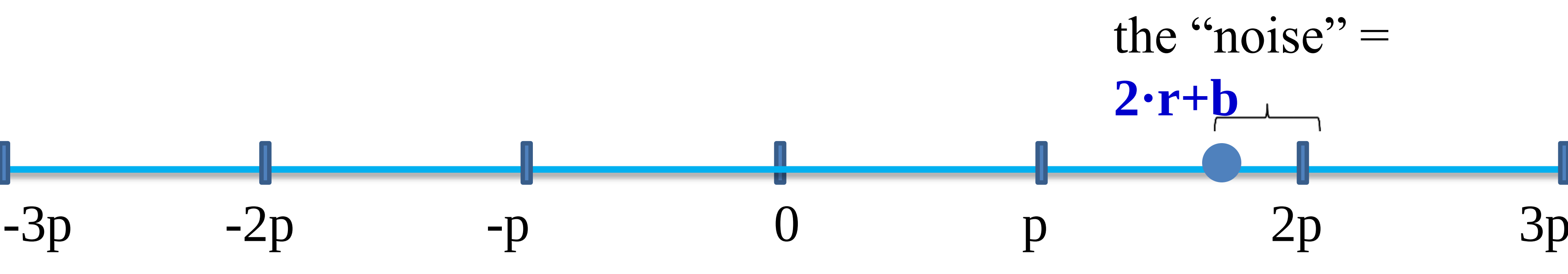
*To Encrypt a bit **b**:*

– pick a (random) “large” multiple of p, say **q·p**

– pick a (random) “small” number **2·r+b**

(this is even if b=0, and odd if b=1)



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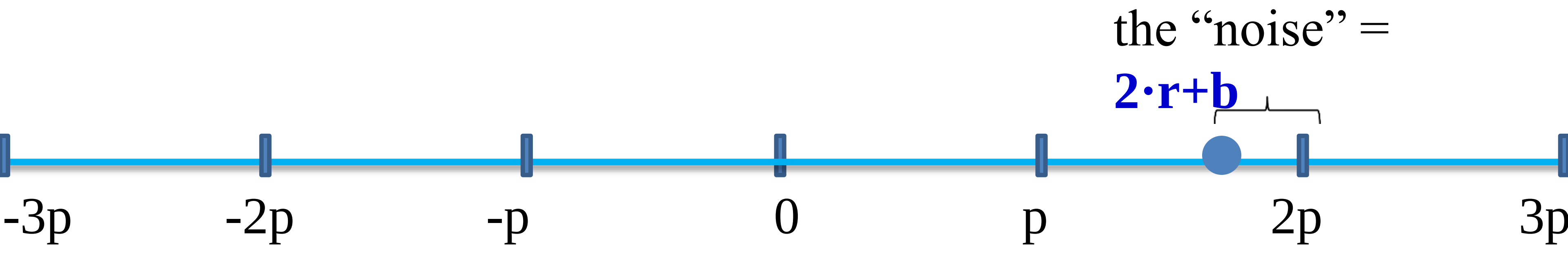
– Ciphertext **c = q·p+2·r+b**



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To Encrypt a bit ***b****:*

– pick a (random) "large" multiple of p, say $\mathbf{q \cdot p}$

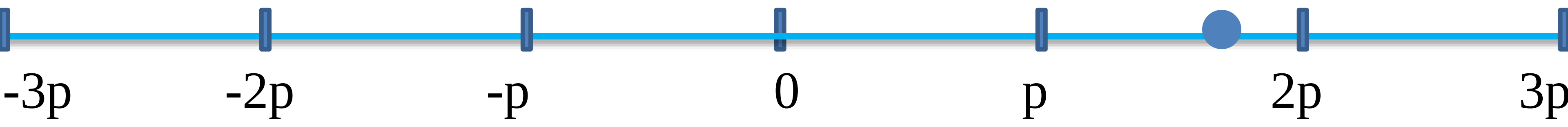
– pick a (random) "small" number $\mathbf{2 \cdot r + b}$

(this is even if b=0, and odd if b=1)

– Ciphertext $\mathbf{c = q \cdot p + 2 \cdot r + b}$

To Decrypt a ciphertext ***c****:*

Taking (*c mod p) mod 2* recovers the plaintext



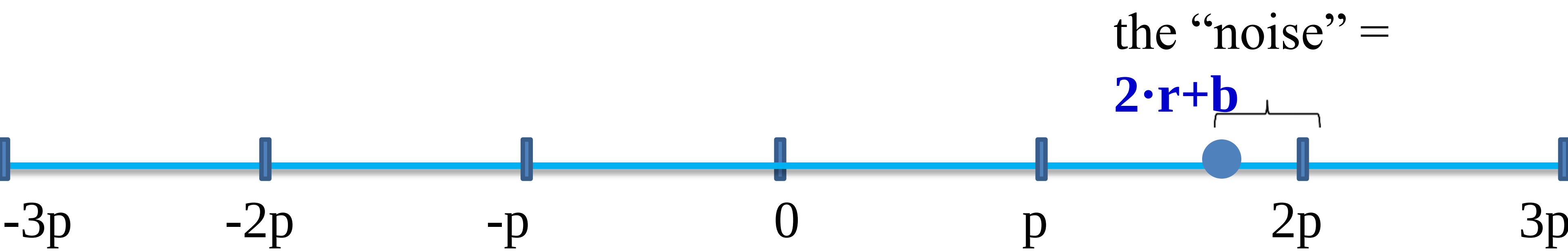
How secure is this?

… if there were no noise (think r=0)

… and I give you two encryptions of 0 (q_1p & q_2p)

… then you can recover the secret key p

$$= \mathrm{GCD}(q_1p,\ q_2p)$$



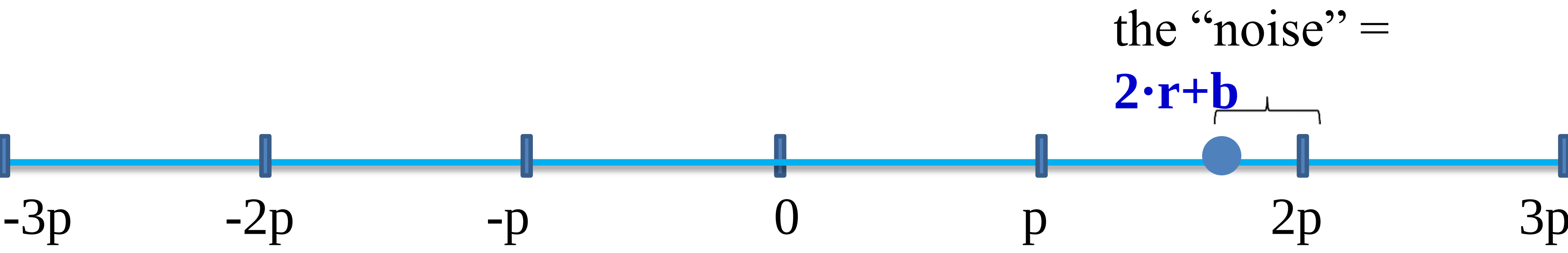
How secure is this?

… but if there is noise

… the GCD attack doesn't work

… and neither does any attack (we believe)

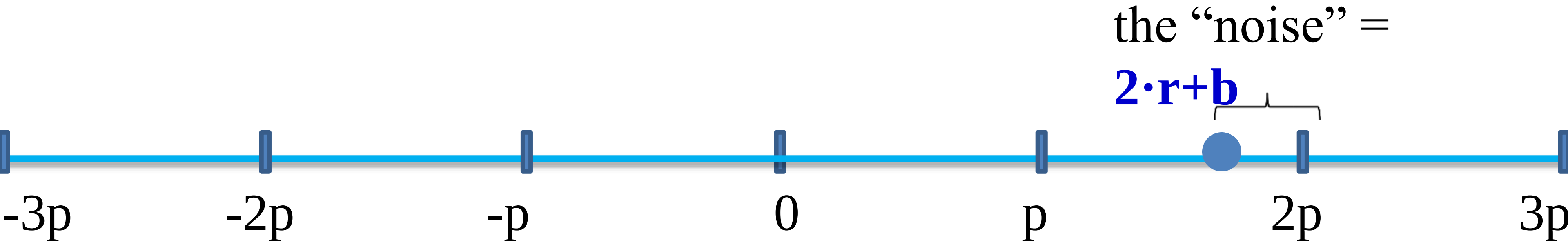
… this is called the *approximate GCD assumption*



XORing two encrypted bits:

– $\mathbf{c_1} = q_1 \cdot p + (2 \cdot r_1 + b_1)$

– $\mathbf{c_2} = q_2 \cdot p + (2 \cdot r_2 + b_2)$



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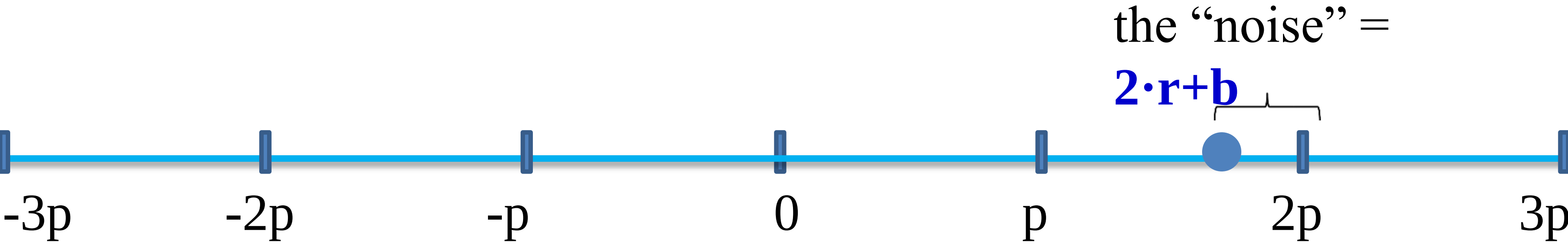

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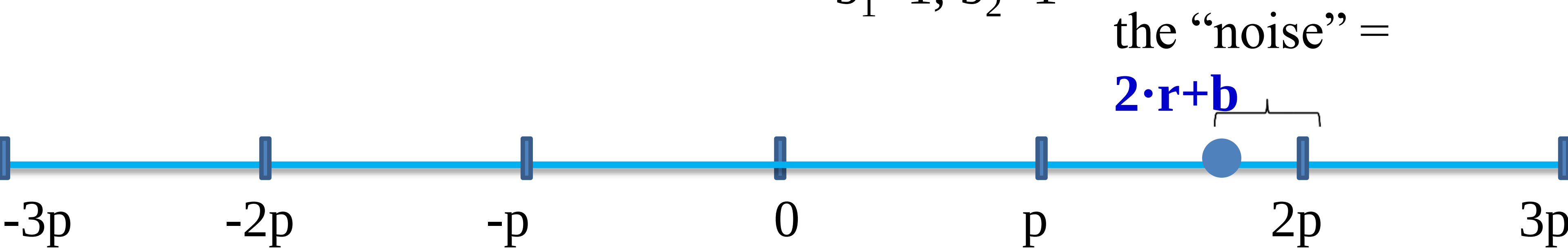
- $\mathbf{c_1 + c_2} = \mathbf{p} \cdot (q_1 + q_2) + \mathbf{2} \cdot (r_1 + r_2) + (b_1 + b_2)$

Odd if $b_1=0, b_2=1$ (or)
$b_1=1, b_2=0$

Even if $b_1=0, b_2=0$ (or)
$b_1=1, b_2=1$



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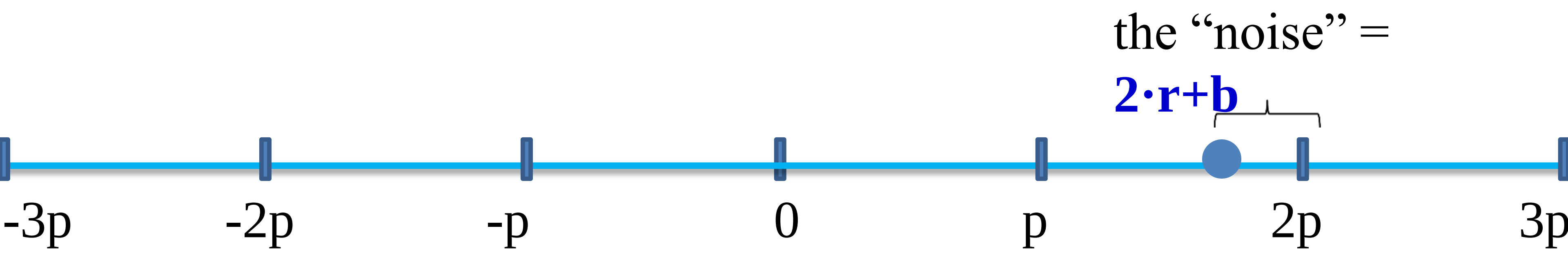
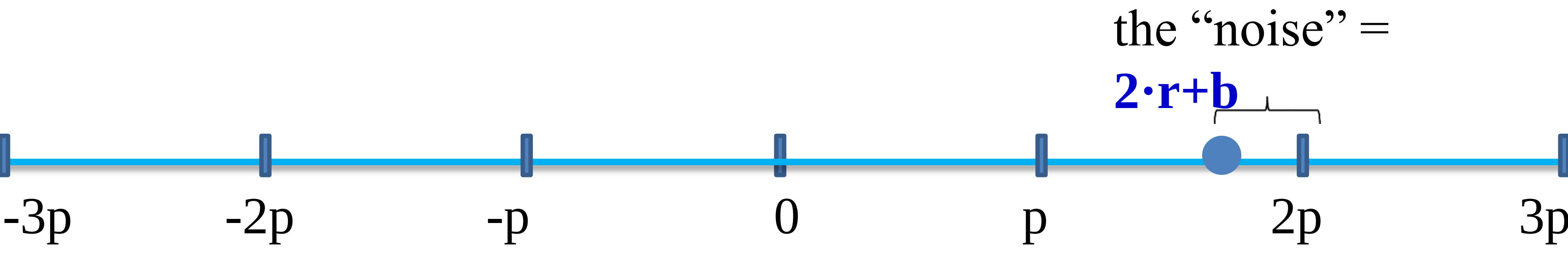
lsb= b_1 XOR b_2



ANDing two encrypted bits:

- $\mathbf{c_1} = q_1 \cdot p + (2 \cdot r_1 + b_1)$
- $\mathbf{c_2} = q_2 \cdot p + (2 \cdot r_2 + b_2)$
- $\mathbf{c_1 c_2} = \mathbf{p \cdot}(c_2 \cdot q_1 + c_1 \cdot q_2 - q_1 \cdot q_2) + \mathbf{2 \cdot}(r_1 r_2 + r_1 b_2 + r_2 b_1) + b_1 b_2$



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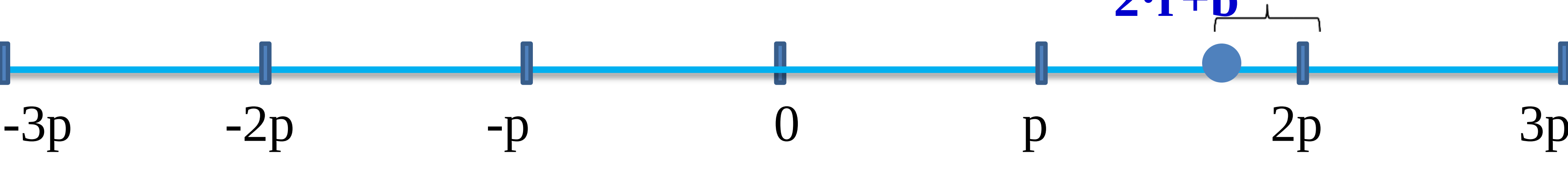
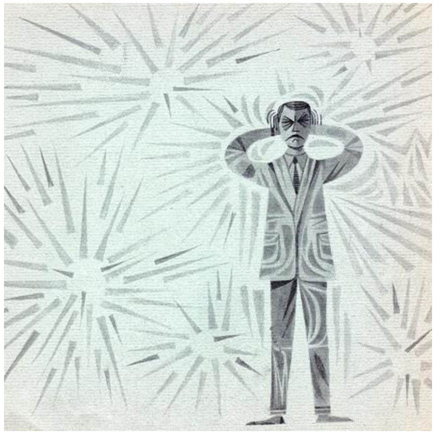
lsb= b_1 AND b_2

the "noise" = **$2 \cdot r + b$**



the noise grows!



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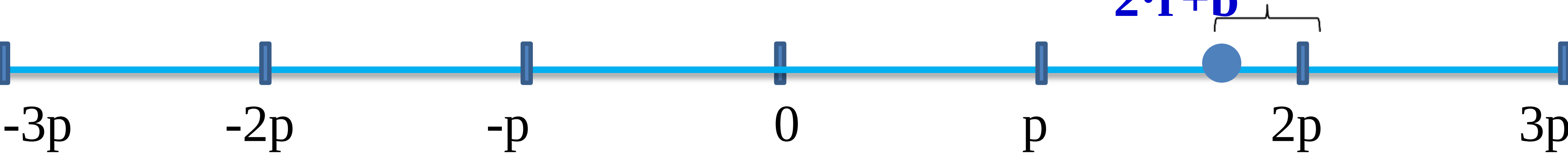

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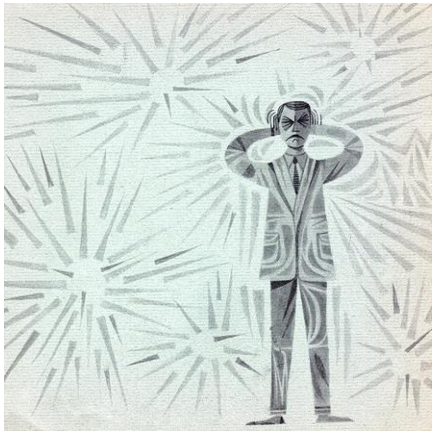
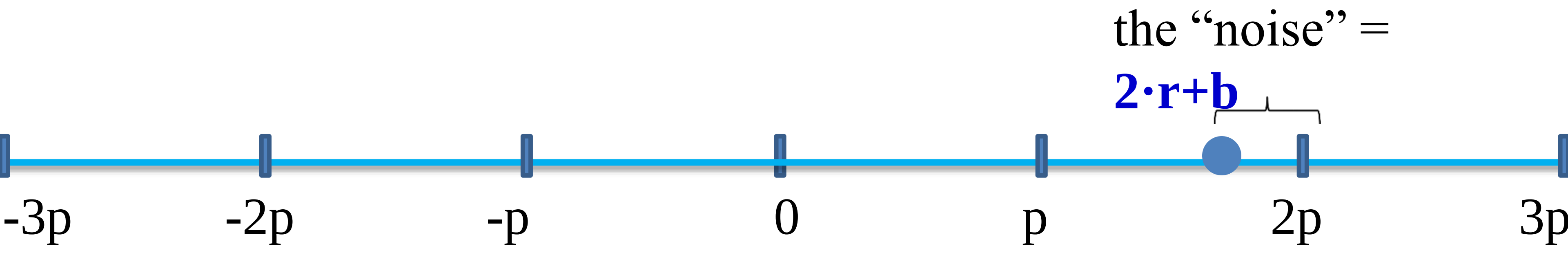


$$c_1+c_2 = p\cdot(q_1 + q_2) + 2\cdot(r_1+r_2) + (b_1+b_2)$$

noise= 2 * (initial noise)



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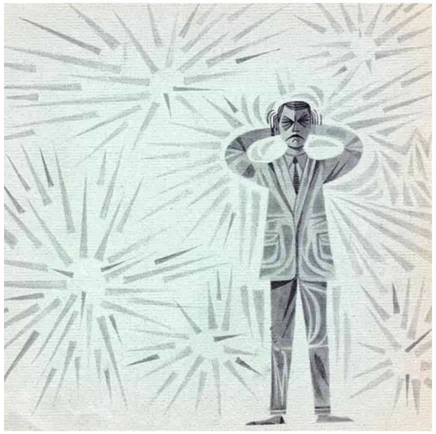

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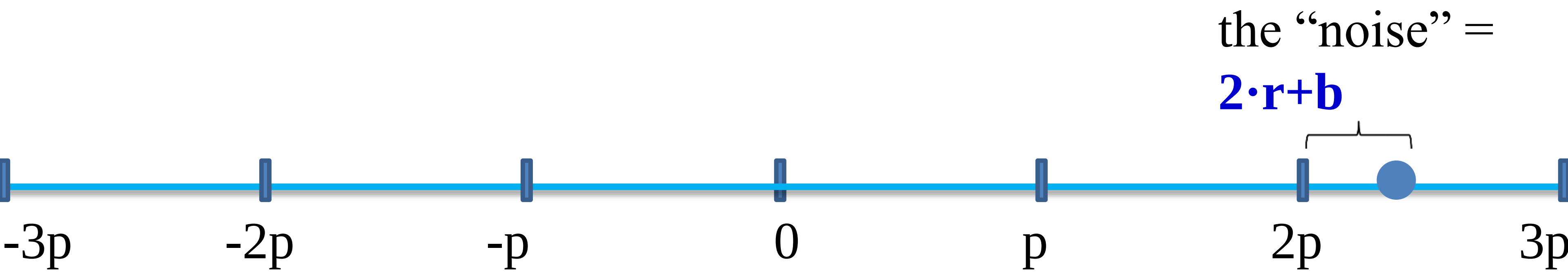
– $\mathbf{c_1 c_2} = \mathbf{p}\cdot(c_2\cdot q_1+c_1\cdot q_2-q_1\cdot q_2) + \mathbf{2}\cdot(r_1r_2+r_1b_2+r_2b_1) + b_1b_2$

noise = (initial noise)^2



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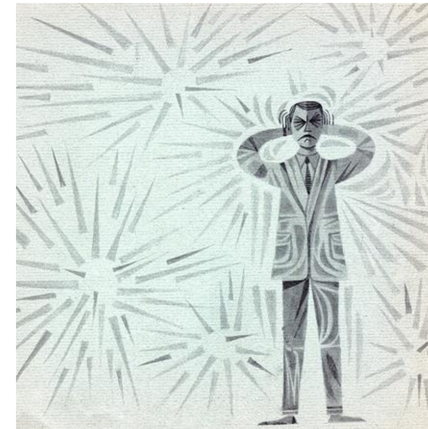
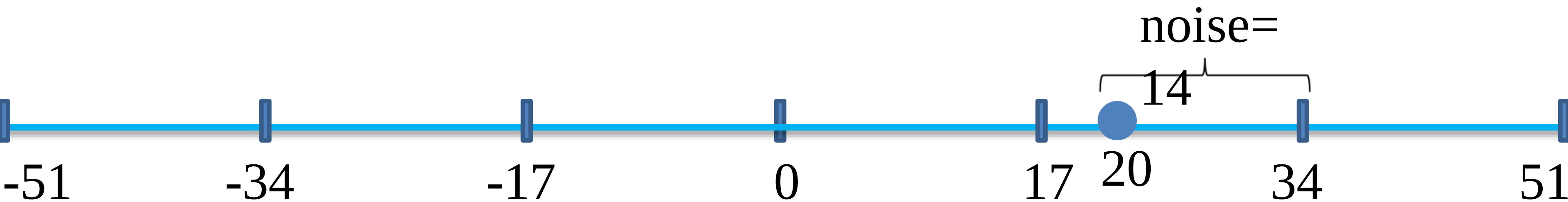


... so what's the problem?



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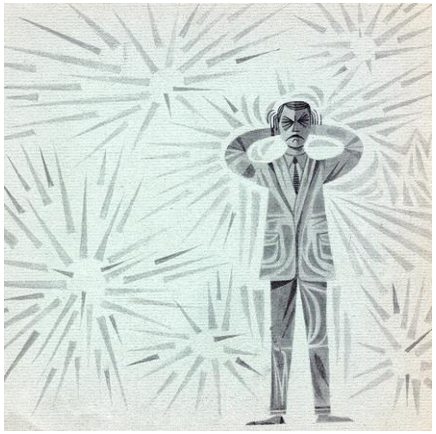

… so what’s the problem?

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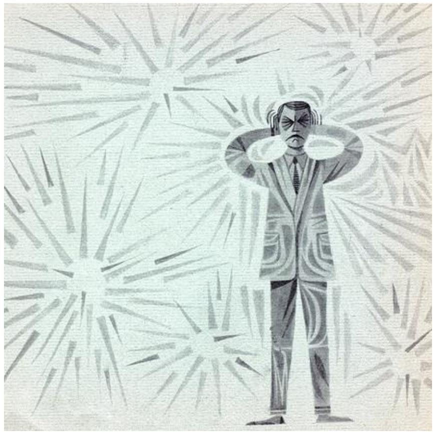

… so what's the problem?

If the |noise| > p, then …

decryption might output an incorrect bit

Example: $E(b) = pq + 2r + b$

If $2r+b<p$ ***then it is always*** $(E(b) \bmod p) \bmod 2 = b$

If $2r+b>p$***, then imagine*** $2r+b=p+1$

If $b=0$***, then*** $(E(b) \bmod p) \bmod 2 = 1$***, not equal to 0***

So, what did we accomplish?

... we can do lots of additions and

... some multiplications

(= a "somewhat homomorphic*" encryption)*

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... enough to do many useful tasks, e.g., database search, spam filtering etc.

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... some multiplications

(= a "somewhat homomorphic" encryption)

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But I promised much more ...


Josh's system
Boneh,Goh & Nissim
Fully homomorphic
MANY add
ZERO mult
MANY add
1 mult
WE ARE HERE!
MANY add
MANY mult

Gentry's ***"bootstrapping theorem" …***

Gentry's ***"bootstrapping theorem"*** *…*

… If you can go a (large) part of the way,

then you can go all the way.

Gentry's ***"bootstrapping theorem"*** *…*

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then you can go all the way.

[HOW? WE'LL SEE IN A BIT]

How efficient is all this?

... can I buy a homomorphic encryption software and start encrypting my data?

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... encrypting a bit takes ~19s (!) with the current best implementation

... it takes 99 min to encrypt this sentence

How efficient is all this?

... can I buy a homomorphic encryption software and start encrypting my data?

... well, not quite yet

... encrypting a bit takes ~19s (!) with the current best implementation

... but we are improving rapidly...

References:

[1] "Computing arbitrary functions of Encrypted Data",
Craig Gentry, Communications of the ACM 53(3), 2010.

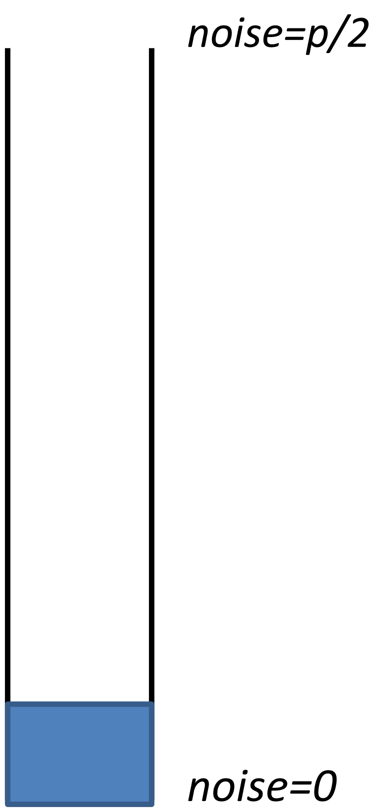
[2] "Fully Homomorphic Encryption from the Integers", Marten van Dijk, Craig Gentry, Shai Halevi, Vinod Vaikuntanathan
http://eprint.iacr.org/2009/616, Eurocrypt 2010.

[3] "Implementing Gentry's Fully Homomorphic Encryption", Craig Gentry and Shai Halevi
https://researcher.ibm.com/researcher/files/us-shaih/fhe-implementation.pdf, Eurocrypt 2011.

Gentry's ***"bootstrapping method"*** ***…***

… If you can go a (large) part of the way,

then you can go all the way…



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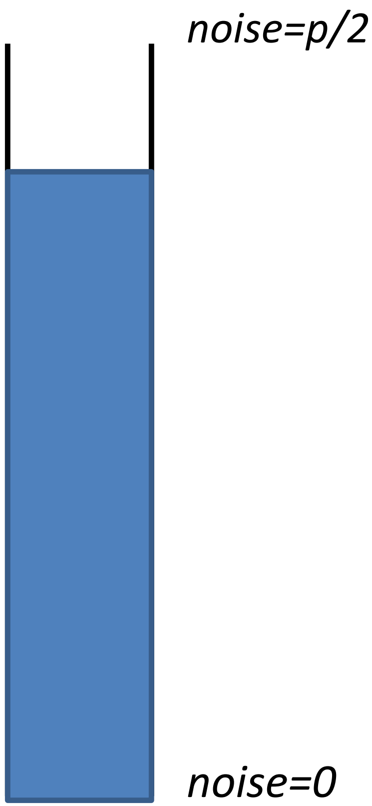

Problem: Add and Mult increase noise

(Add doubles, Mult squares the noise)

Gentry's ***"bootstrapping method" …***

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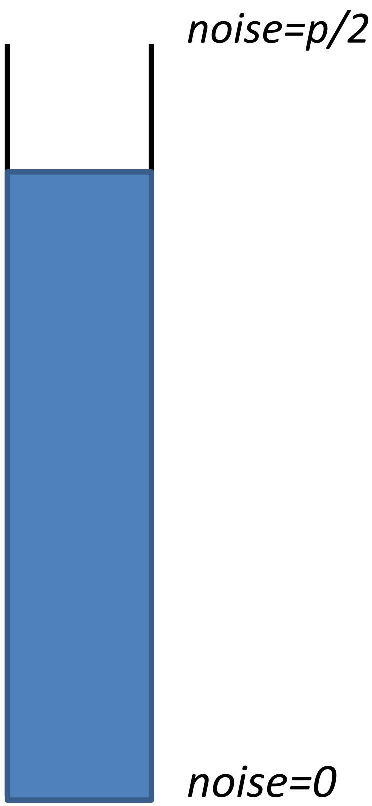


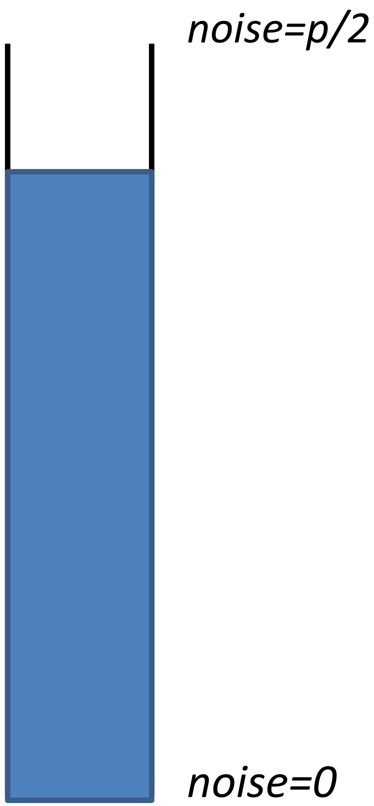
Problem: Add and Mult increase noise

(Add doubles, Mult squares the noise)

So, we want to do *noise-reduction*

Let's think…

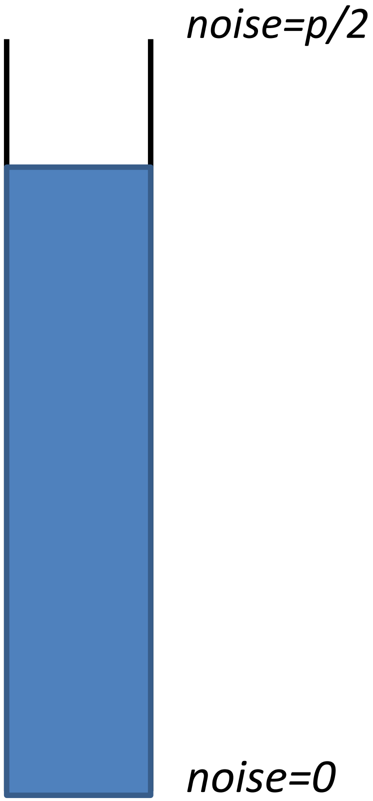
… What is the best noise-reduction procedure?



Let's think…

… What is the best noise-reduction procedure?

… something that kills all noise

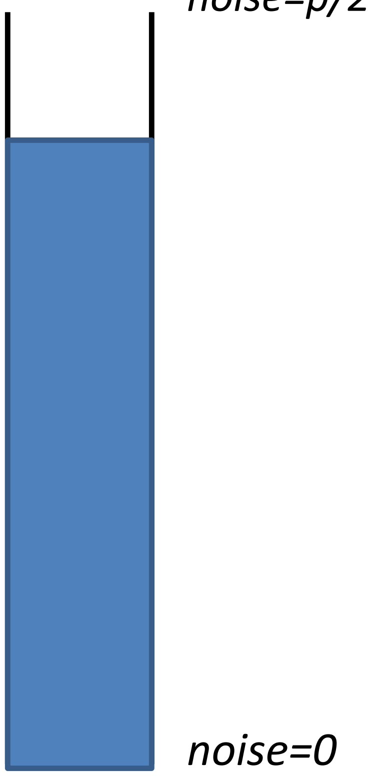


Let's think…

… What is the best noise-reduction procedure?

… something that kills all noise

… and recovers the message



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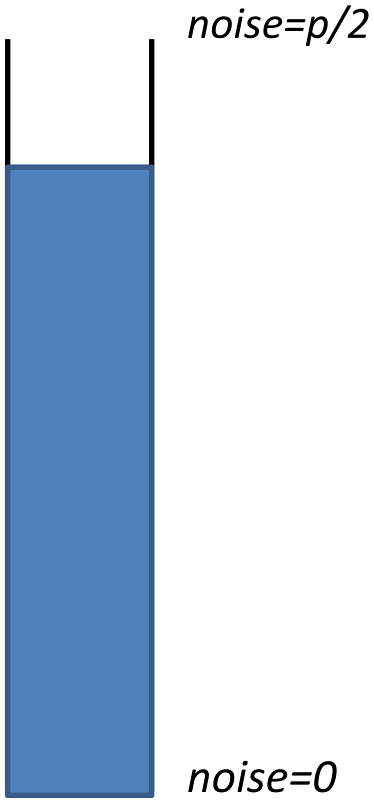

Decryption!

Let's think…

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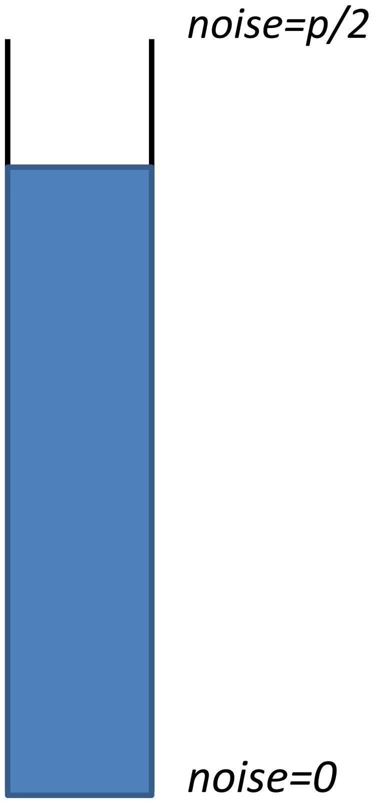
… and recovers the message



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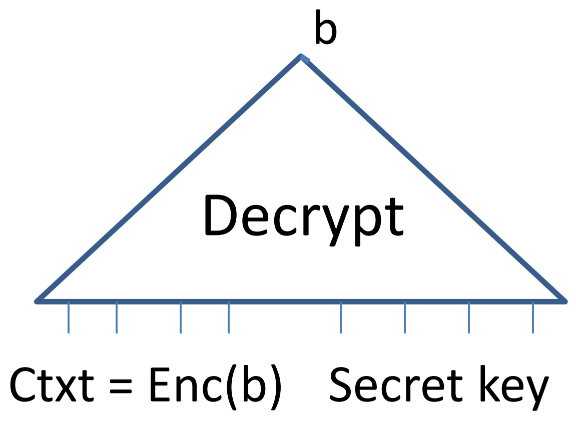

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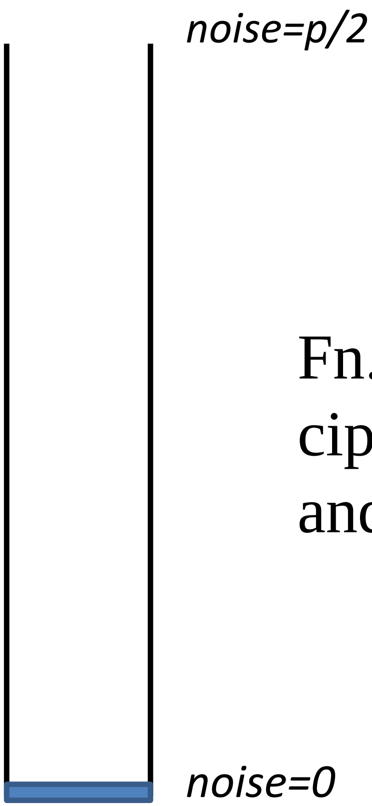
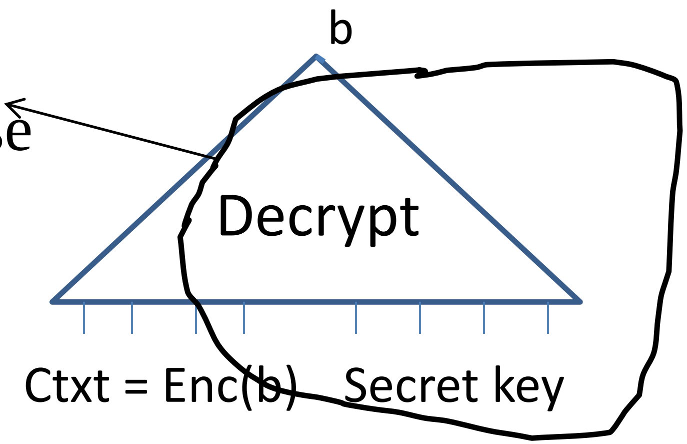


… and recovers the message

Decryption!

Fn. that acts on ciphertext
and eliminates noise



Let's think…

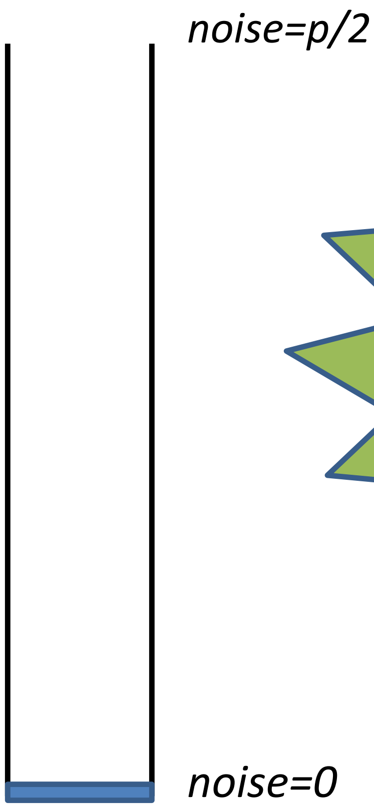
… What is the best noise-reduction procedure?

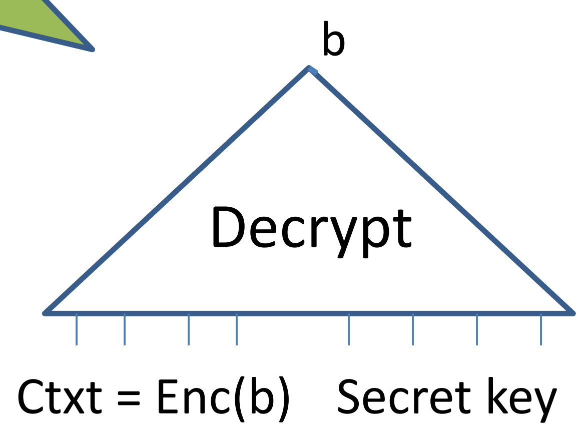
… something that kills all noise

… and … covers the message

Decryption!

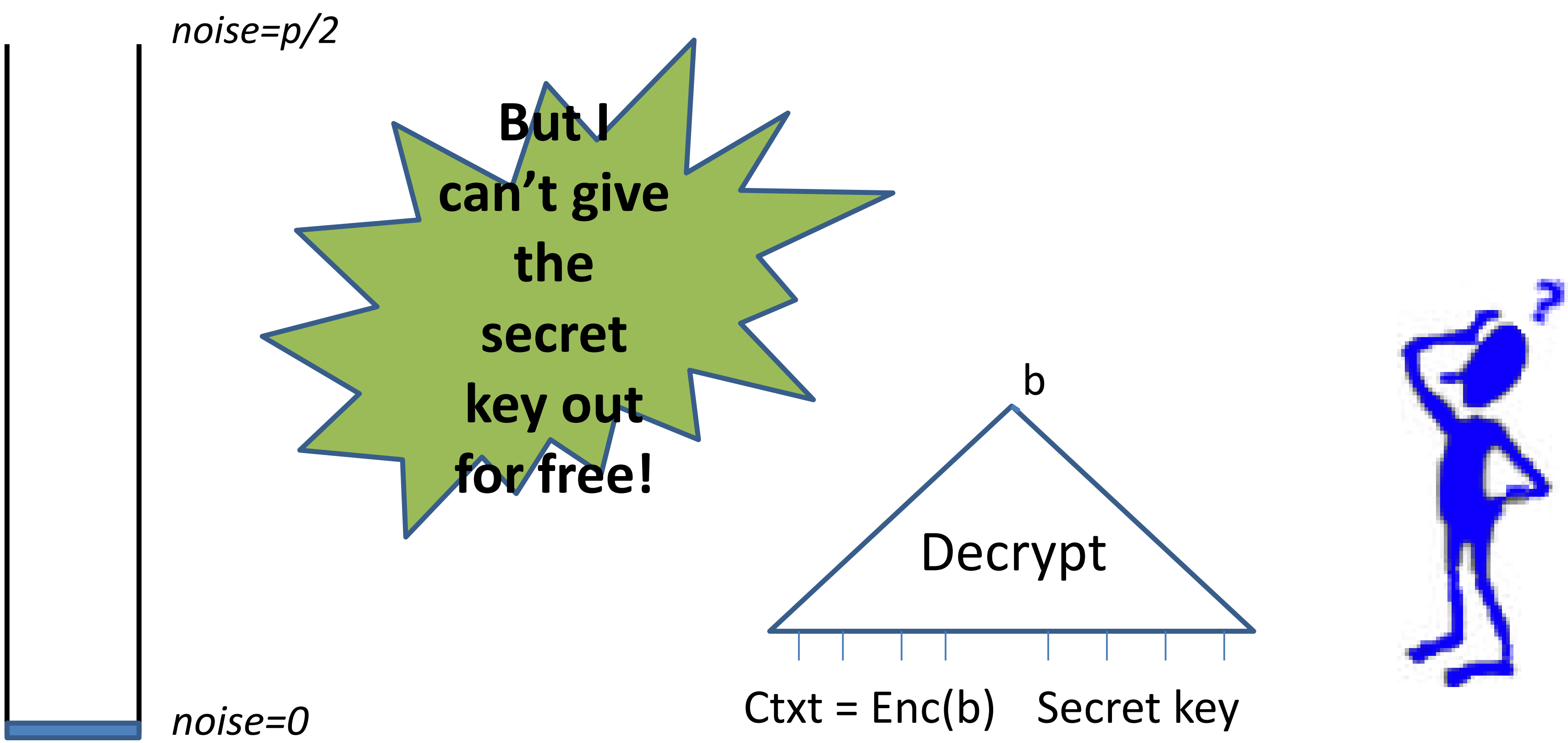
But I can't give the secret key out for free!





Let's think…

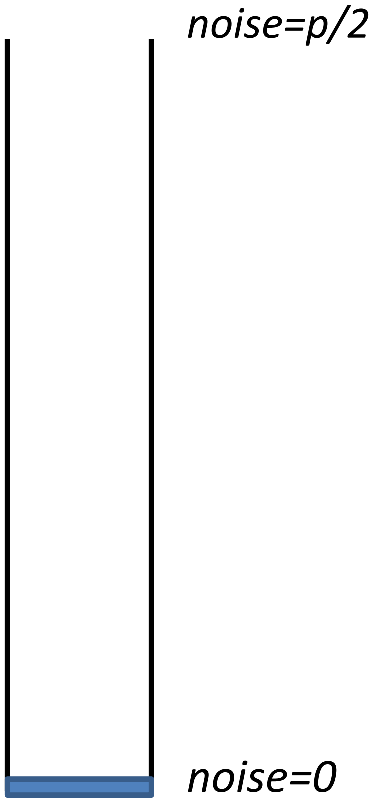
… I want to reduce noise *without letting you decrypt*

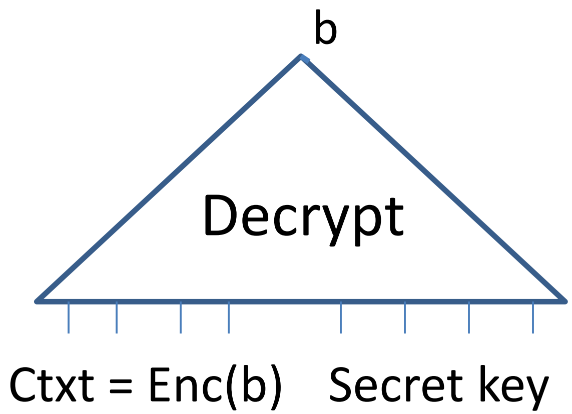


KEY IDEA:

… I cannot release the secret key (lest everyone sees my data)

… but *I can release Enc(secret key)*





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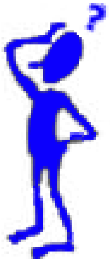
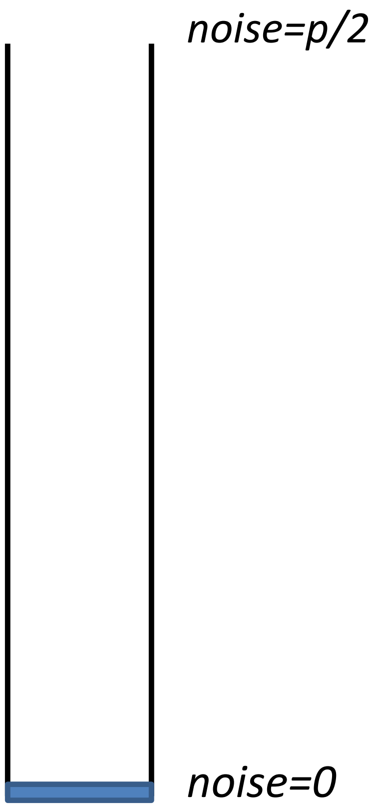
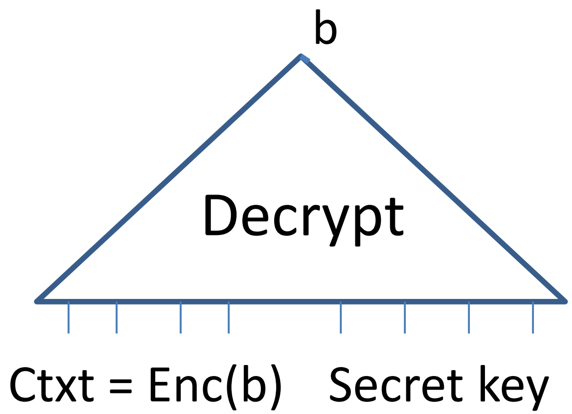
… called "Circular Encryption"







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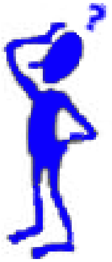

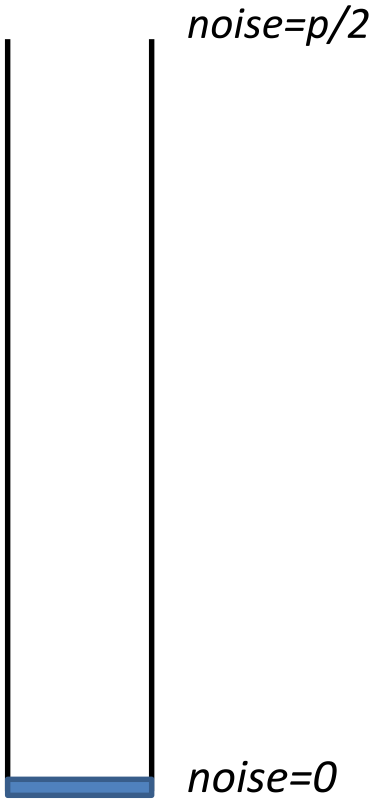

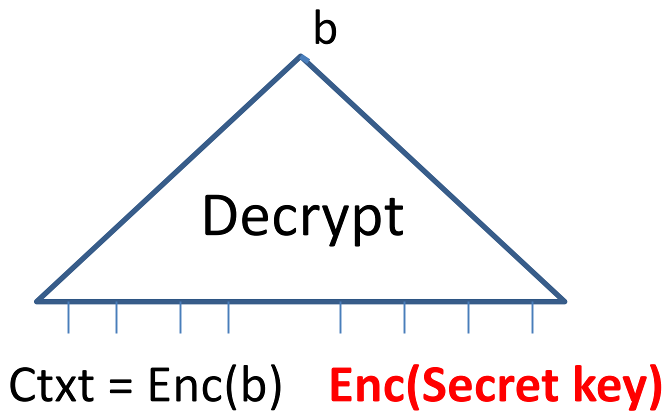

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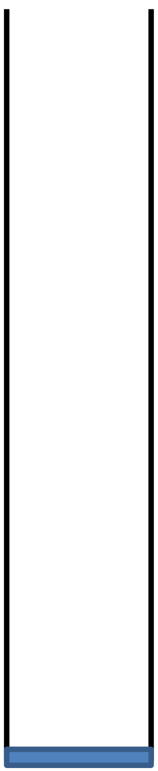
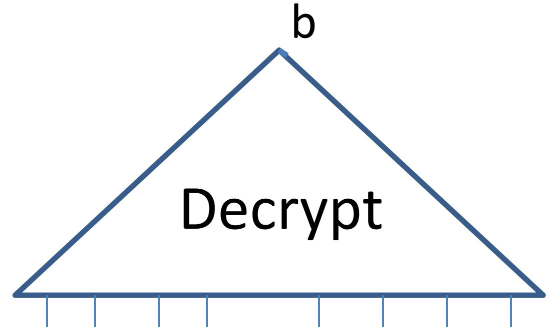


… Now, to reduce noise …

… Homomorphically evaluate the decryption ckt!!!



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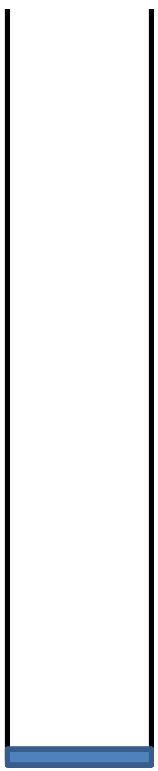

… Now, to reduce noise …

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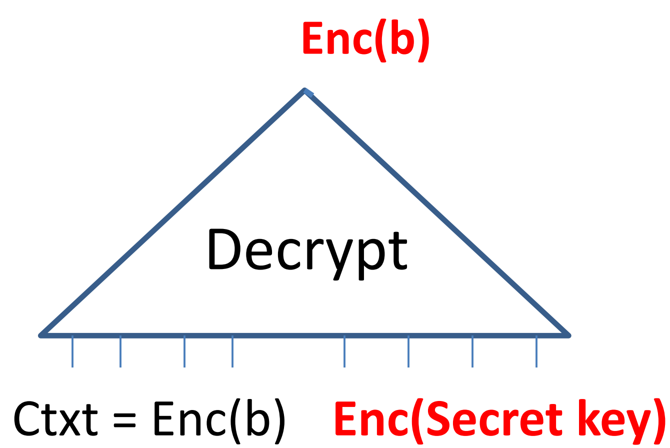

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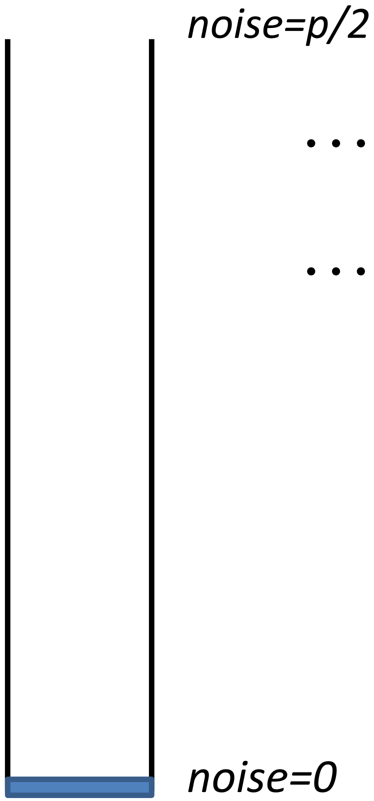

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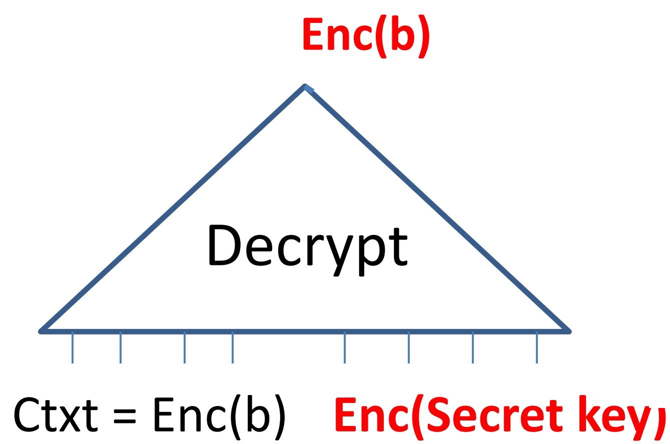

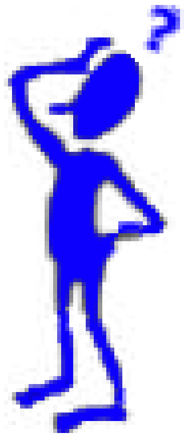

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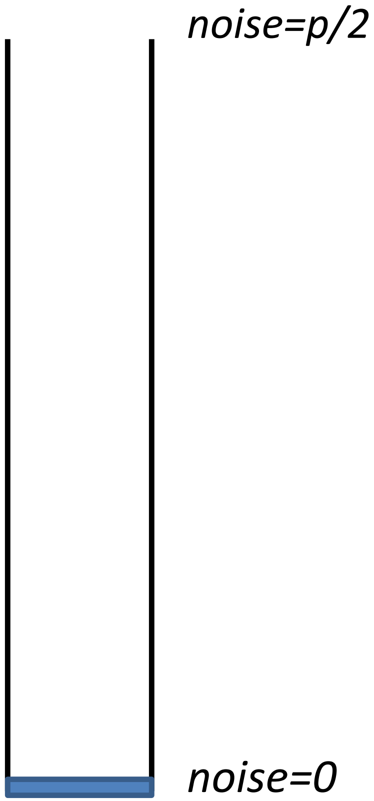
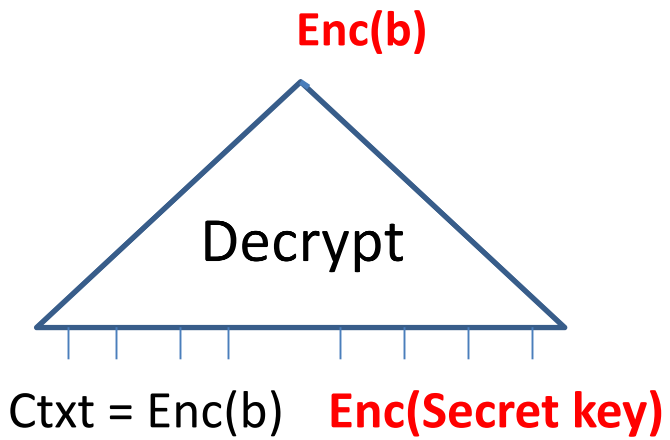


KEY OBSERVATION:

… the input Enc(b) and output Enc(b) have different noise levels …



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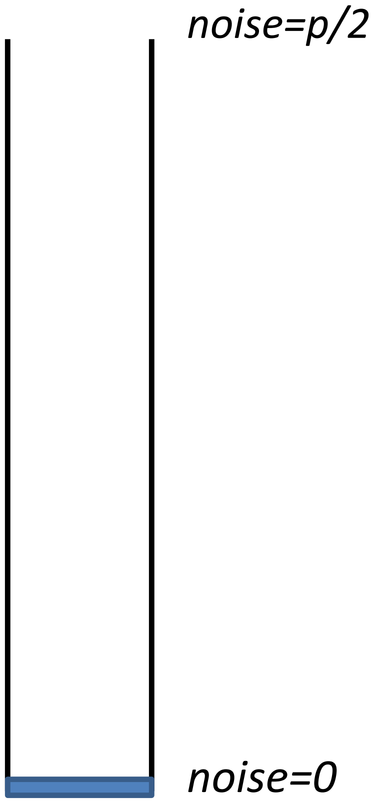
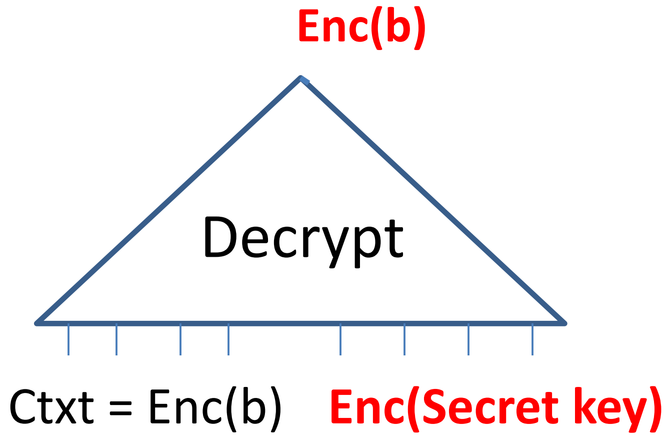


KEY OBSERVATION:

Regardless of the noise in the input Enc(b)…

the noise level in the output Enc(b) is **FIXE**



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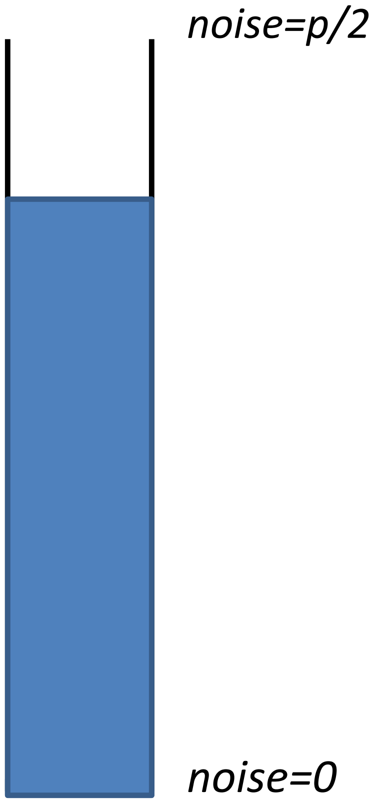

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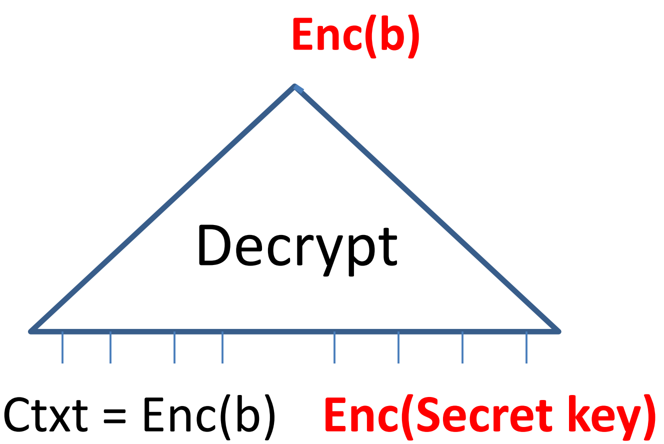

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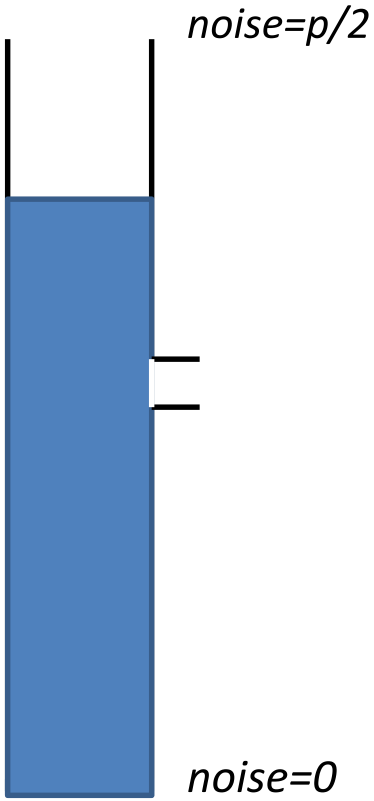

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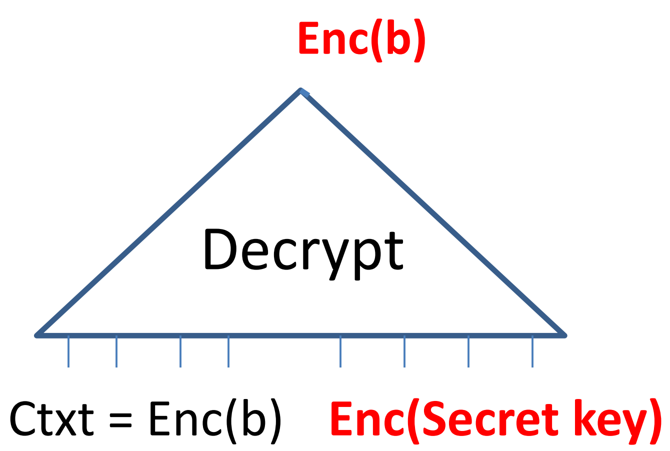

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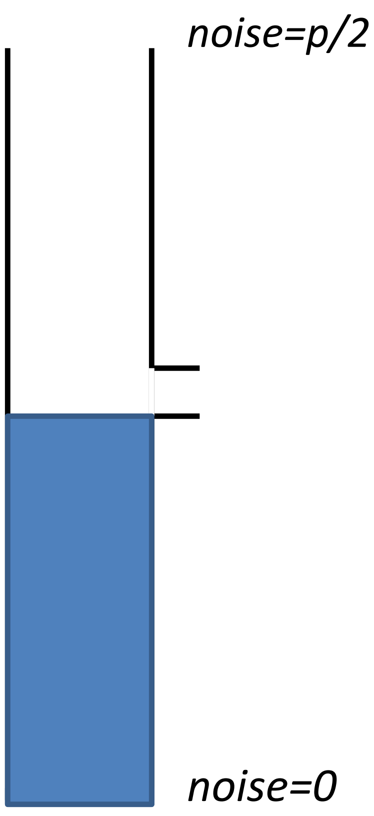

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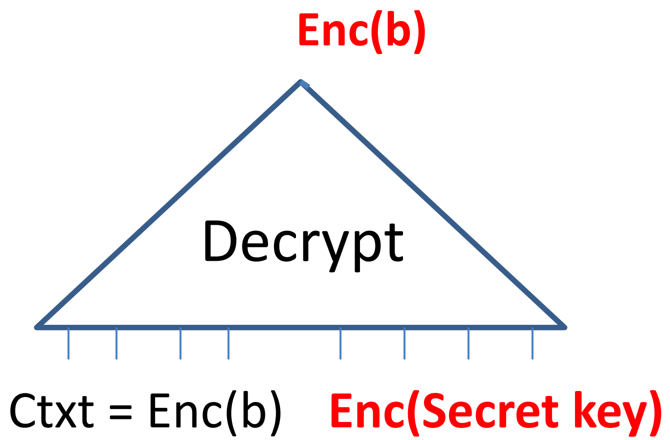
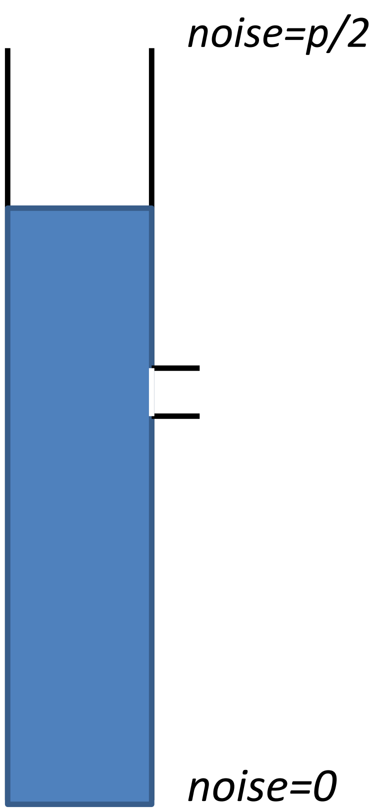
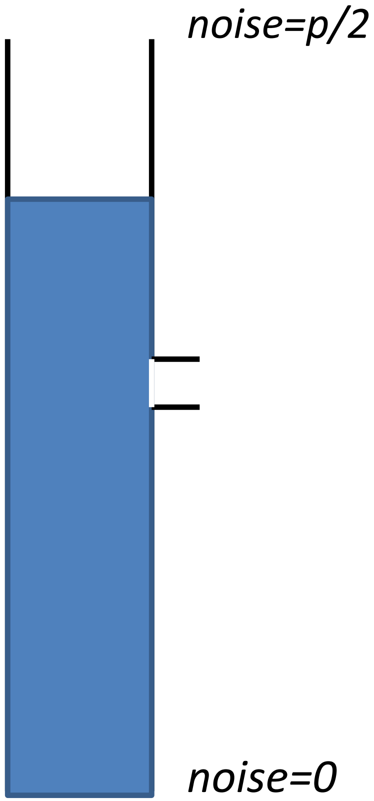


Bottomline: *whenever noise level increases beyond a limit …*

… use bootstrapping to reset it to a fixed level



Bootstrapping requires homomorphically evaluating the decryption circuit …



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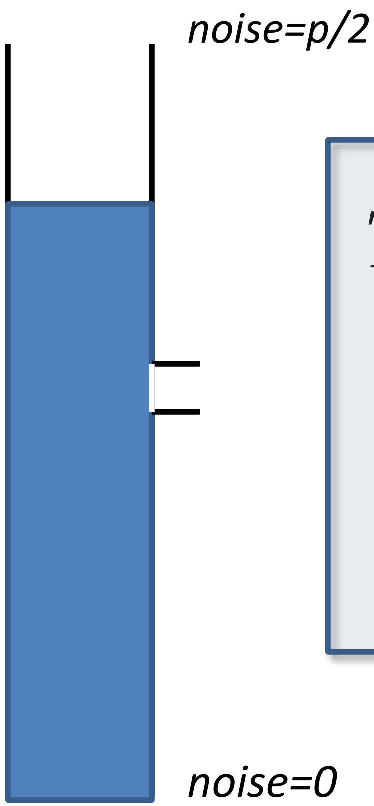

Thus, Gentry's ***"bootstrapping theorem":***

If an enc scheme can evaluate its own decryption circuit, then it can evaluate everything